Chain and Hamilton - Jacobi approaches for systems with purely second class constraints

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Abstract

The equivalence of the chain method and Hamilton-Jacobi formalism is demonstrated. The stabilization algorithm of Hamilton-Jacobi formalism is clariffied and two examples are presented in details.

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1 Introduction

The quantization of second-class constrained systems, initiated by Dirac [1] and developed by various authors [2] is subjected to intense debates during the last years. A powerful method, based on the introduction of extra variables was proposed in [3] and it leads us to convert the second class-constraints into first class ones. Recently, in [4] a second-class constrained system was investigated and a gauge theory was found without introducing extra variables. In [5] the authors proposed to eliminate half of the second-class constraints and converted the second half to the first-class constraints. This method is called the chain-method and it was subjected recently to various investigations [6] In the chain method one primary constraint is producing, under some assumptions, a set of constraints.At the end all primary constraints produce a chain of second class constraints but some of them are in involution. The next step of the method is to produce a gauge theory by adding an extra term in the canonical Hamiltonian in such a manner that half of the constraints are eliminated [5].Despite of many attempts to elucidate the integrability conditions of the second-class constrained systems in Hamilton- Jacobi formalism (HJ) based on $Carath\acute{e}odory$'s approach [7] and initiated in [8] there are some hidden parts which must be clarified. The main problem is to clarify when the stabilization algorithm come to an end and to make a deep exploration of the equivalence with Dirac's formulation. On the other hand we have to keep in mind the physical interpretation of (HJ) partial differential equations involved in (HJ) formalism [9]. All this issues are important for the construction of the integrability of the second-class constrained systems in (HJ) formalism [10].

The main aim of this paper is to show that the stabilization algorithms produce the same constraints and that the lagrange multipliers of chain method arise naturally in (HJ) formalism.

The plan of the paper is as follows:

In sec. 2 the (HJ) formalism and the chain method are briefly presented. In sect. 3 the equivalence of both method are investigated and two examples were analyzed. Sec. 4 is devoted to conclusions.

2 The methods

2.1 Hamilton-Jacobi formalism

Let us assume that the Lagrangian L is singular and the Hessian supermatrix has rank n-r. The Hamiltonians to start with are

$$H_{\alpha}^{'} = H_{\alpha}(t_{\beta}, q_a, p_a) + p_{\alpha}, \tag{1}$$

where $\alpha, \beta = n-r+1, \cdots, n$,$a = 1, \cdots n-r$. The usual Hamiltonian H_0 is defined as

$$H_0 = -L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a) + p_a w_a + \dot{q}_\mu p_\mu \mid_{p_\nu = -H_\nu}, \nu = 0, n-r+1, \cdots, n. \quad (2)$$

which is independent of $\dot{q}_\mu$.Here $\dot{q}_a = \frac{dq_a}{d\tau}$,where τ is a parameter. The equations of motion are obtained as total differential equations in many variables as follows

$$\begin{aligned} dq_a &= (-1)^{P_a + P_a P_\alpha} \frac{\partial_r H_\alpha^{'}}{\partial p_a} dt_\alpha, dp_a = -(-1)^{P_a P_\alpha} \frac{\partial_r H_\alpha^{'}}{\partial q_a} dt_\alpha, \\ dp_\mu &= -(-1)^{P_\mu P_\alpha} \frac{\partial_r H_\alpha^{'}}{\partial t_\mu} dt_\alpha, \mu = 1, \cdots, r, \end{aligned} \quad (3)$$

$$dz = (-H_\alpha + (-1)^{P_a + P_a P_\alpha} p_a \frac{\partial_r H_\alpha^{'}}{\partial p_a}) dt_\alpha, \quad (4)$$

where $z = S(t_\alpha, q_a)$ and P_i represents the parity of a_i.

On the surface of constraints the system of differential equations(3) is integrable if and only if

$$[H_\alpha^{'}, H_\beta^{'}] = 0, \forall \alpha, \beta. \quad (5)$$

2.2 The chain method

Let us consider a singular Lagrangian $L(q_a, \dot{q}_a), a = 1, \cdots N$ admitting a primary constraint

$$\phi_1(p_a, q_a) = 0. \quad (6)$$

We assume that the Lagrangian possesses only $2n(n < N)$ second-class constraints. The total Hamiltonian is

$$H = H_c + \lambda \phi_1, \quad (7)$$

where H_c is the canonical Hamiltonian and λ is a parameter to be determined. Taking into account the integrability conditions we obtain the following one-chain system:

$$\phi_2(p_a, q_a) \equiv \{\phi_1, H\} = 0, \cdots, \phi_{2n}(p_a, q_a) \equiv \{\phi_{2n-1}, H\} = 0. \quad (8)$$

Since the set of constraints ϕ_i is composed of second-class constraints, the matrix $M_{ij} = \{\phi_i, \phi_j\}$ is nonsingular. Explicitly,

$$M_{ij} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -\alpha \\ 0 & 0 & 0 & \cdots & 0 & \alpha & * \\ 0 & 0 & 0 & & -\alpha & * & * \\ \vdots & & & & & & \vdots \\ 0 & 0 & \alpha & & 0 & * & * \\ 0 & -\alpha & * & & * & 0 & * \\ \alpha & * & * & \cdots & * & * & 0 \end{pmatrix}, on \quad \Sigma_{2n}. \quad (9)$$

Here "on $\sum_{2n}$" means the weak equality modulo the constraints $\phi_1 = \phi_2 = \cdots = \phi_{2n} = 0$. On the other hand we have

$$\{\phi_1, \phi_n\} = -\alpha(p_a, q_a) \tag{10}$$

and half of the second class constraints, as we can be seen from (9), are in involution [5].The advantage of this method is that it gives some new information about the form of M_{ij} and that some of the constraints are in involution. The next step is to generalize the procedure to a chain of K primary constraints. We denote these constraints by $\phi_1^{(1)}, \phi_1^{(2)}, \cdots, \phi_1^{(K)}$.

$$\begin{array}{cccc}
\phi_1^{(1)} = 0 & \phi_1^{(2)} = 0 & \cdots & \phi_1^{(K)} = 0 \\
\phi_2^{(1)} \equiv \{\phi_1^{(1)}, H\} = 0 & \phi_2^{(2)} \equiv \{\phi_1^{(2)}, H\} = 0 & \cdots & \phi_2^{(K)} \equiv \{\phi_1^{(K)}, H\} = 0 \\
\phi_{M_1}^{(1)} \equiv \{\phi_{M_1-1}^{(1)}, H\} = 0 & \phi_{M_2}^{(2)} \equiv \{\phi_{M_2-1}^{(2)}, H\} = 0 & \cdots & \phi_{M_K}^{(K)} \equiv \{\phi_K^{(K)}, H\} = 0
\end{array} \tag{11}$$

The Hamiltonian H is given by

$$H = H_c + \sum_{m=1}^{K} v_0^{(m)} \phi_1^{(m)} \tag{12}$$

$$\{\phi_i^{(l)}, \phi_j^{(m)}\} \approx 0 \quad , i \leq r_l, \quad j \leq r_m; l, m \quad arbitrary \quad , \tag{13}$$

Using the same arguments as before we observe that the $K \times K$ matrix

$$\alpha \equiv \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1K} \\ \vdots & & \vdots \\ \alpha_{K1} & \cdots & \alpha_{KK} \end{pmatrix} \tag{14}$$

where $\alpha_{mn} = \{\phi_{r_l+1}^{(l)}, \phi_{r_m}^{m}\}$ must have at least one non-zero element in each row [5] The method is applicable to local field theory too. Let us assume that the local Lagrangian density L is singular and admits only one second-class constraints $\phi_1(\vec{x}), \phi_2(\vec{x}), \cdots, \phi_{2n}(\vec{x})$ and only $\phi_1(\vec{x})$ is primary. In this method we have infinite number of constraint equations. If we calculate the bracket of constraints we will construct the matrix $M_{ij} = \{\phi_i, \phi_j\}$ as

$$M_{ij}(\vec{x}, \vec{y}) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & -\alpha \\
0 & 0 & 0 & \cdots & 0 & \alpha & * \\
0 & 0 & 0 & & -\alpha & * & * \\
\vdots & & & & & & \vdots \\
0 & 0 & \alpha & & 0 & * & * \\
0 & -\alpha & * & & * & 0 & * \\
\alpha & * & * & \cdots & * & * & 0
\end{pmatrix} \delta(\vec{x} - \vec{y}) \tag{15}$$

To find the subset of constraints which are in involution we follow the procedure used before.To simplify the problem and to explain the method we consider

the two-chain case. Let us assume the set of the constraints in involution as $\{\chi_1, \chi_2, \cdots, \chi_r; \psi_1, \psi_2, \cdots, \psi_q\}$. To stop the chain we transform the Hamiltonian as

$$H^{''} = H_c + \frac{1}{2}\chi^T \alpha^{-1} \chi, \tag{16}$$

where

$$\chi = \begin{pmatrix} \chi_{r+1} \\ \psi_{q+1} \end{pmatrix} \tag{17}$$

For field theory let us assume that only a primary constraint Φ_1 generates 2n-1 constraints denoted by $\Phi_\alpha, \alpha = 2, \cdots 2n$. Using the same procedure as before we construct the corresponding extended Hamiltonian as

$$H^{'''} = H_c + \frac{1}{2}\int d\vec{x}\alpha^{-1}(\vec{x})\Phi^2_{n+1}(\vec{x}), \tag{18}$$

such that we eliminated half of the constraints and the resulting system is a first-class one.

3 Equivalence of the methods

Let us consider a singular Lagrangian L admitting 2n second class constraints and K of them are primary. Let us denote the primary constraints by $H_1^{'}, H_2^{'} \cdots, H_K^{'}$. Consistency conditions read as

$$dH_\alpha^{'} = \{H_\alpha^{'}, p_0 + H_0\}d\tau + \{H_\alpha^{'}, H_\beta^{'}\}dt_\beta, \quad \alpha, \beta = 1, \cdots K. \ , \tag{19}$$

$H_0 = H_c$ There are two cases to be studied. First, all of the variations (19)might vanish identically. In this case there is no need to go one further step. The system is integrable.Second, some of the variations might vanish identically, and the other variations, say m of them, do not vanish. These non-vanishing variations will give (m) differential equations in $\dot{t}_\beta = \frac{dt_\beta}{d\tau}$. Here one should notice that although we impose the conditions that t_β are independent variables, theory forces us to calculate them as the solutions of differential equations which arise from the variations. Now, since all variations are zero the system is integrable. The price we pay for this is the determination all of the independent variables as function of τ. As we can see from (19), when $\tau = t$, $\dot{t}_\beta$ is nothing that the expression of the Lagrange multiplier from chain corresponding to the primary constraint H_β. This result is valid in general , at the final stage , all velocities of the gauge variables are the Lagrange multipliers of the chain method. The next step in the (HJ) formalism is to calculate the action. Since the system is second-class, the "Hamiltonians" are not in involution. One way to solve this problem is to try to solve all equations of motion.When we are dealing with non-linear equations this way is very difficult to be used. Another way to bypass this problem is to make the "Hamiltonians" in involution by selecting half of them and transforming $H_0^{'}$ as in (16) for discrete case or as in (18) for fields.

3.1 Examples

To illustrate the approaches we will present two examples.

1.Consider the Lagrangian

$$L=\frac{1}{2}(\dot{q}_1+q_5)^2+\frac{1}{2}(\dot{q}_2+q_6)^2+\frac{1}{2}(\dot{q}_3^2+\dot{q}_4^2)-q_5(q_2+V_1(q_3,q_4))+q_6(q_1+V_2(q_3,q_4))-V_3(q_3,q_4), \tag{20}$$

where V_1, V_2 and V_3 are any functions of their arguments. Let us apply (HJ) formalism for this system. The primary constraints are

$$H_1^{'} \equiv p_5, H_2^{'} \equiv p_6 \tag{21}$$

The usual canonical Hamiltonian is

$$H_0=\frac{1}{2}\sum_{i=1}^{4}{p_i}^2-q_5(p_1-q_2-V_1(q_3,q_4))-q_6(p_2+q_1+V_2(q_3,q_4))+V_3(q_3,q_4), \tag{22}$$

so

$$H_0^{'}=p_0+H_0 \tag{23}$$

These expressions give the following equations of motion:

$$\begin{array}{cc} dq_1=(p_1-q_5)dt, & dq_2=(p_2-q_6)dt, dq_3=p_3dt, dq_5=dq_5, \\ dq_4=p_4dt, & dp_1=q_6dt, dp_2=-q_5dt, dq_6=dq_6 \\ dp_3=(-q_5\frac{\partial V_1}{\partial q_3}+q_6\frac{\partial V_2}{\partial q_3}-\frac{\partial V_3}{\partial q_3})dt & dp_4=(-q_5\frac{\partial V_1}{\partial q_4}+q_6\frac{\partial V_2}{\partial q_4}-\frac{\partial V_3}{\partial q_4})dt. \end{array} \tag{24}$$

Using (24) variations of constraints $H_1^{'}$ and $H_2^{'}$ follows as

$$dH_1^{'}=\{H_0^{'},H_1^{'}\}dt+\{H_2^{'},H_1^{'}\}dq_6, dH_2^{'}=\{H_0^{'},H_2^{'}\}dt+\{H_1^{'},H_2^{'}\}dq_5, \tag{25}$$

so

$$H_3^{'}=p_1-q_2-V_1, H_4^{'}=p_2+q_1+V_2. \tag{26}$$

Imposing the variations of $H_3^{'}$ and $H_4^{'}$ to be zero and taking into account (26) we obtain another two new "Hamiltonians":

$$H_5^{'}=2q_6-p_2-\frac{\partial V_1}{\partial q_3}p_3-\frac{\partial V_1}{\partial q_4}p_4, H_6^{'}=-2q_5+p_1+\frac{\partial V_2}{\partial q_3}p_3+\frac{\partial V_2}{\partial q_4}p_4. \tag{27}$$

If we consider the variations of (27) we obtain a first order equation for q_5 and q_6. A simple calculation shows that $\dot{q}_5$ and $\dot{q}_6$ have the same form as the Lagrange multipliers from chain method. At this point we conclude that our results are in agreement to [5] . If we choose, for example, $V_1(q_3,q_4)=q_3, V_2(q_3,q_4)=q_4$ and $V_3(q_3,q_4)=0$, then (24) becomes

$$dq_1 = (p_1-q_5)dt, dq_2=(p_2-q_6)dt, dq_3=p_3dt, \tag{28}$$

$$dq_4=p_4dt \quad , dp_1=q_6dt, dp_2=-q_5dt, dp_3=-q_5dt, dp_4=q_6dt. \tag{29}$$

The solution of (28) is

$$
\begin{aligned}
q_6(t) &= C_9\cos(t) - C_{10}\sin(t), q_5(t) = C_9\sin(t) + C_{10}\cos(t), && (30)\\
q_4(t) &= -C_9\cos(t) + C_{10}\sin(t) + C_7 t + C_5, && (31)\\
q_3(t) &= C_9\sin(t) + C_{10}\cos(t) + C_8 t + C_6, q_2(t) = C_4 t + C_1, && (32)\\
q_1(t) &= C_3 t + C_2, p_1(t) = C_9\sin(t) + C_{10}\cos(t) + C_3, && (33)\\
p_2(t) &= C_9\cos(t) - C_{10}\sin(t) + C_4, p_3(t) = C_9\cos(t) - C_{10}\sin(t) + C_8, && (34)\\
p_4(t) &= C_9\sin(t) + C_{10}\cos(t) + C_7, && (35)
\end{aligned}
$$

where $C_i, i = 1, \cdots 10$ are constants.

Imposing $H_3^{'} = H_4^{'} = H_5^{'} = H_6^{'} = 0$ we obtain the following restrictions on the above constants

$$C_3 = C_7, C_4 = -C_8, C_4 = -C_2 - C_5, C_3 = C_1 + C_6. \tag{36}$$

As it is seen the above example is two-chain example and three constraints are commuting each other. The main aim was to make the system integrable and to find the action.One way is to introduce the expressions of $q_i, i = 1 \cdots 6$ in $H_0^{'}$ and then we will find the action using only one "Hamiltonian". In the following we will apply the other method, mainly we will use only half of the constraints and we will try to modify the Hamiltonian $H_0^{'}$ such that it will be involution with them. Having in mind to obtain an integrable system and with physical interpretation from (HJ) point of view we choose the following "Hamiltonians"

$$H_1^{'} = p_5, H_2^{'} = p_6, H_3^{'} = p_1 - q_2 - V_1 \tag{37}$$

Using (16) the new form of $H_0^{'}$ is

$$
\begin{gathered}
H_0^{''} = p_0 + \tfrac{1}{2}(p_1^2 + p_2^2 + p_4^2) + \tfrac{1}{8}(p_2 - q_1 - V_2)^2 + V_3\\
-\tfrac{1}{2}(p_2 + q_1 + V_2)(2q_6 - p_2 - \tfrac{\partial V_1}{\partial q_3}p_3 - \tfrac{\partial V_1}{\partial q_4}p_4) + \tfrac{1}{8}(p_2 + q_1 + V_2)^2(1 + (\tfrac{\partial V_1}{\partial q_3})^2 + (\tfrac{\partial V_1}{\partial q_4})^2)
\end{gathered}
\tag{38}
$$

Since $H_1^{'}, H_2^{'}, H_0^{''}$ are commuting the corresponding system is integrable.
The corresponding action is

$$S = \int dt\{\frac{1}{2}(p_3^2 + p_4^2) + \frac{1}{8}[{p_2}^2 - (q_1 + V_2)^2][1 + (\frac{\partial V_1}{\partial q_3})^2 + (\frac{\partial V_1}{\partial q_4})^2]\} \tag{39}$$

2. The bosonized Lagrangian for the chiral model in (1+1) dimensions is written as [11]

$$L = \int dx[\frac{1}{2}(\partial_\mu\phi + A_\mu)(\partial^\mu\phi + A^\mu) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \varepsilon^{\mu\nu}\partial_\mu\phi A_\nu] \tag{40}$$

The canonical Hamiltonian density becomes

$$H_c = \int dx[\frac{1}{2}(\pi^2 + (\partial_x\phi)^2 + \pi_1^2) + \pi_1\partial_x A_0 + (\pi + \partial_x\phi + A_1)(A_1 - A_0)], \quad (41)$$

where π, π_0, π_1 represent the momenta conjugate to ϕ, A_0, A_1 respectively. In (HJ) formalism the primary Hamiltonian density is

$$H_1^{'} = \pi_0. \quad (42)$$

The equations of motions corresponding to (41) and (42) are

$$\begin{array}{c} d\phi = (\pi + A_1 - A_0)dt, dA_1 = (\pi_1 + \partial_x A_0)dt, \\ d\pi = [\partial_x(A_1 - A_0) + \partial_x^2\phi]dt, d\pi_1 = -(2A_1 + A_0 - \phi - \partial_x\phi)dt \end{array} \quad (43)$$

Taking into account the variation of (42) we obtain the second "Hamiltonian" as

$$H_2^{'} = \partial_x(\pi_1 + \phi) + \pi + A_1 \quad (44)$$

From (44) we obtain another "Hamiltonian" density as

$$H_3^{'} = \pi_1 \quad (45)$$

Finally, imposing the variation of $H_3^{'}$ to be zero we obtain the last "Hamiltonian" density as

$$H_4^{'} = -\pi - \partial_x\phi - 2A_1 + A_0 \quad (46)$$

We identified the matrix M_{ij} from chain method (15) as

$$M(x,y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix} \delta(x-y). \quad (47)$$

As we can easily see from (47) the "Hamiltonians" $H_1^{'}, H_2^{'}, H_3^{'}, H^{'}$ are not in involution and using (5)the system of equations (43)is not integrable. In addition, we mention that if we continue to consider the variation of $H_4^{'}$ we will obtain that $\dot{A}_0$ is nothing the value of the Lagrange multiplier in the Dirac's formulation. The modified canonical density Hamiltonian becomes

$$H_0^{''} = p_0 + H_c - \frac{1}{2}\pi_1^2. \quad (48)$$

We have three "Hamiltonian" densities which are in involution

$$\begin{array}{c} H_0^{''} = p_0 + \frac{1}{2}(\pi^2 + (\partial_x\phi)^2) + \pi_1\partial_x A_0 + (\pi + \partial_x\phi + A_1)(A_1 - A_0), \\ H_1^{'} = \pi_0, H_2^{'} = \pi + \partial_x(\pi_1 + \phi) + A_1 \end{array} \quad (49)$$

If we ignore the surface term, the action corresponding to (49) becomes

$$S = \int dxdt[-\frac{1}{2}(\pi^2 + (\partial_x\phi)^2) - (\pi + \partial_x\phi + A_1)(A_1 - A_0) - \dot{\phi}A_1]. \quad (50)$$

4 Conclusions

In this paper we proved that the chain method and (HJ) formalism are equivalent. In other words both stabilization algorithms gave the same set of constraints. Since the system is second-class constrained it appears that the last Hamiltonian or Hamiltonian density in (HJ) contains a variable which was undefined by the system of the equations corresponding to the primary constraints and the Hamiltonian $H_0^{'}$. If we take the variation of the last Hamiltonain we obtained the velocity of that variable. We have shown that this velocity is nothing that the Lagrange multiplier in Dirac's formalism. In fact, in (HJ) formalism we have no Lagrange multipliers to start with but we have a set of variables q_α corresponding to the primary constraints $H_\alpha^{'} = p_\alpha + H_\alpha$, $\alpha = 1, \cdots K$. And the end of the stabilization algorithm of (HJ) we obtained a set of "Hamiltonians" and a set of new equations for $\dot{q}_\alpha$. The remaining problem is to analyzed the integrability of the system of total differential equations. For second class-constrained systems (HJ) formalism is not integrable in the sense of (5). To make it integrable we have several options corresponding to several techniques of modifying the set of second class-constraints into a first-class one. For (HJ) formalism this step is crucial because the action delivered by it depends drastically of the form of the "Hamiltonians". In other words only for "Hamiltonians" in the form $H_\alpha^{'} = p_\alpha + H_\alpha$ it is possible to calculate the action given by (4). Following this idea we calculated the action of (HJ) corresponding to the gauge system obtained after applying the chain method.

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